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DYNAMIC BEHAVIOR OF STRUCTURES COMPOSED OF STRAIN AND WORKHARDENING VISCO-PLASTIC MATERIALS

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Abstract—A dynamic convergence thereon is proven for a class of visco-plastic constitutive equations involving internal state variables, which provides an extension of a result due to Martin[1]. The class of constitutive equation, which includes the Malvern material when effective plastic strain is adopted as the state variable, is expressed in terms of a flow potential. For a simple model structure both convergence and divergence is demonstrated. The example also demonstrates that the theorem provides a sufficient but not a necessary condition for convergence, as convergence can occur even when the conditions of the theorem are not satisfied.

1. INTRODUCTION

During the last ten years approximate techniques have been developed by Martin, Symonds and others[1–5] which provide solutions to the problem of a structure subjected to an initial impulse. The technique employed, the "mode" solution method, derives from an observation by Martin[1], that structures subjected to differing initial distributions of velocity have convergent velocity fields during the deformation process. Under some circumstances the velocities may converge to a "mode" solution which is provided by the solution for a particular distribution of initial velocities, and in other circumstances the "mode" solutions are defined by the extremal of a functional of the velocity distribution[3, 6]. The technique has been developed for large deformation of structures[5], and provides a simple direct method of assessing the most significant features of the deformation of impulsively loaded structures and obviates the need for a complete analysis.

The theory has, however, been developed only for constitutive relationships for which either the inelastic strain or strain rate is given in terms of the instantaneous stress. Although a wide class of material models fall within this category (elastic, perfectly plastic, deformation theory plastic, visco-rigid plastic and viscous materials), the important class of work and strain hardening viscoplastic models are excluded.

The purpose of the paper is to present extensions of the convergence proof of Martin[1] to a class of state variable equations which may be expressed in terms of a dissipation potential. This class has been discussed by Ponter[7] for quasi-static loading of a body. For uniaxial stress σ with resulting elastic strain e and inelastic strain v, the constitutive relationship take the form

$$\dot{\epsilon} = \dot{e} + \dot{v}, \quad \dot{e} = G\dot{\sigma} \tag{1}$$

$$\dot{v} = \frac{\partial}{\partial \sigma}\Omega(\sigma, S) \tag{2}$$

$$\dot{S} = -\frac{\partial}{\partial S}\Omega(\sigma, S) \tag{3}$$

where ϵ denotes the total strain, G the elastic compliance and Ω a potential function of stress and a state variable S. By choosing a particular state variable, eqn (2) and (3) yield a specific functional form for the potential Ω. For example, we may choose

$$S = l(v) \tag{4}$$

where l is some arbitrary function, and restrict stress histories to these for which $\sigma > 0$ and v

increases in time. Eliminating $\dot{v}$ between (2) and (3) and using (4) results in the differential equation

$$\frac{\partial \Omega}{\partial \sigma} + \frac{1}{l'^2} \frac{\partial \Omega}{\partial v} = 0. \tag{5}$$

The general solution of (5) is given by

$$\Omega(\sigma, S) = \Omega(\sigma - g(v)) \tag{6}$$

where

$$g(v) = \int l'^2 \, \mathrm{d}v \tag{7}$$

Subsitituting this solution into (2) yields

$$\dot{v} = \Omega'(\sigma - g(v)) \tag{8}$$

which may be recognized as Malvern's[8] relationship where $g(v)$ is the static yield stress corresponding to monotonic flow to inelastic strain v, and $\sigma - g(v)$ is the dynamic overstress.

Hence the eqn (2) and (3) provided a limited but important class of constitutive relationship, and we propose to show that this class can possess convergence properties for dynamically loaded body if Ω is a convex function of its arguments.

In Section 2 some results from [7] are reviewed and the inversion of the constitutive relationship is discussed. In Section 3 three particular cases are described, isotropic and kinematic strain hardening and work hardening visco-plastic models. In Section 4 the convergence theorem is formally derived for the general class. This is followed by Section 5 where a simple structure is analyzed for situations where sufficient conditions for the convergence theorem are both satisfied and violated. This is achieved by varying the material parameters in a simple strain hardening visco-plastic constitutive relationship. In particular, the conditions are violated for materials whose static plastic stress strain curve has increasing slope with strain. For sufficiently pronounced strain hardening characteristics nonconvergence is exhibited by the model although we also find cases where convergence occurs.

2. A CLASS OF CONSTITUTIVE RELATIONS

We are concerned with the small strain behavior of a material with inelastic strain rates which are expressed in terms of a simple potential function $\Omega(\sigma_{ij}, S_l)$ where σ_{ij} is the current stress and S_l is the current value of a set of state variables. The total strain ϵ_{ij} is expressed as the sum of a linear elastic strain e_{ij} and inelastic strain v_{ij}.

$$\epsilon_{ij} = e_{ij} + v_{ij} \tag{9}$$

$$e_{ij} = C_{ijhk}\sigma_{hk} \tag{10}$$

and the inelastic strain rate $\dot{v}_{ij}$ and the state rates $\dot{S}_l$ are expressed in terms of Ω as:

$$\dot{v}_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}}, \tag{11a}$$

$$\dot{S}_l = -\frac{\partial \Omega}{\partial S_l}, \tag{11b}$$

The quantities S_l may form a set of scalar quantities or the components of tensor quantities with respect to some fixed axis. The eqn (11) defines a restricted class of consitutive relationships which certainly contain some well known forms [7], but excludes others. We are exploiting the particular strong properties of (11) which allow extension of known results to a wider, but incomplete, class of constitutive relationships.

Before discussing particular forms of eqn (11), we review properties of Ω derived elsewhere. In [7] it was shown that Ω is a convex function of both σ_{ij} and S_l provided the following inequalities hold: at constant state a small change in $d\sigma_{ij}$ results in a corresponding change $d\dot{v}_{ij}$ so that

$$d\sigma_{ij}d\dot{v}_{ij} \geq 0, \quad \dot{S}_l = 0; \tag{12}$$

and for constant σ_{ij}, a small change in state dS_l produces a change in $\dot{S}_l$ such that

$$dS_l d\dot{S}_l \leq 0, \quad \dot{\sigma}_{ij} = 0. \tag{13}$$

From (12) and (13) the convexity of Ω may be derived,

$$\Omega(\sigma^1_{ij}, S^1_l) - \Omega(\sigma^1_{ij}, S^2_l) - \left(\frac{\partial \Omega}{\partial \sigma_{ij}}\right)_2 (\sigma^1_{ij} - \sigma^2_{ij}) - \left(\frac{\partial \Omega}{\partial S_l}\right)_2 (S^1_l - S^2_l) \geq 0. \tag{14}$$

Reversing the roles of (σ^1_{ij}, S^1_l) and (σ^2_{ij}, S^2_l) in (14) and adding the resulting inequality to (14) results in the inequality;

$$(\sigma^1_{ij} - \sigma^2_{ij})(\dot{v}^1_{ij} - \dot{v}^2_{ij}) - (S^1_l - S^2_l)(\dot{S}^1_l - \dot{S}^2_l) \geq 0. \tag{15}$$

Equality in (14) and (15) may occur for stresses and states at which equality occurs in (12) and (13) for non-zero $d\sigma_{ij}$ and dS_l. In inequality (12) such conditions would occur in the rigid region for material with an initial yield surface and in (13) for states of stress and states for which the state is frozen.

The sign of inequality (13) is influenced by the notion of strain hardening; the rate of increase of the state quantities tends to decrease as the state variables increase. If the inequality were reversed then Ω would be convex in σ_{ij} and concave in S_l.

The eqn (11) may be inverted without difficulty. We define dual potentials by

$$\Omega = \int_0^{\sigma_{ij}} \dot{v}_{ij} d\sigma_{ij}, \quad \bar{\Omega} = \int_0^{\dot{v}_{ij}} \sigma_{ij} d\dot{v}_{ij}, \quad \dot{S}_l = 0, \tag{16}$$

so that

$$\Omega(\sigma_{ij}, S_l) + \bar{\Omega}(\dot{v}_{ij}, S_l) = \sigma_{ij}\dot{v}_{ij}. \tag{17}$$

From (17) we see that

$$\left(\frac{\partial \Omega}{\partial S_l}\right)_{\sigma_{ij}} = -\left(\frac{\partial \bar{\Omega}}{\partial S_l}\right)_{\dot{v}_{ij}} \tag{18}$$

and hence from (11), (16) and (18),

$$\sigma_{ij} = \left(\frac{\partial \bar{\Omega}}{\partial \dot{v}_{ij}}\right)_{S_l}, \tag{19a}$$

$$\dot{S}_l = \left(\frac{\partial \bar{\Omega}}{\partial S_l}\right)_{\dot{v}_{ij}} \tag{19b}$$

By analogous arguments to those used for deriving the convexity of Ω we conclude that $\bar{\Omega}$ is convex in $\dot{v}_{ij}$ but concave in S_l.

$$\bar{\Omega}(\dot{v}^2_{ij} S^2_l) - \bar{\Omega}(\dot{v}^1_{ij}, S^1_l) - \left(\frac{\partial \bar{\Omega}}{\partial \dot{v}_{ij}}\right)_1 (\dot{v}^2_{ij} - \dot{v}^1_{ij}) - \left(\frac{\partial \bar{\Omega}}{\partial S_l}\right)_2 (S^2_l - S^1_l) \geq 0. \tag{20}$$

It is worth noting that if inequality (13) was reversed then $\bar{\Omega}$ would be convex in $\dot{v}_{ij}$ and S_l. Hence the reversal on inequality (13) interchanges the convexity properties of Ω and $\bar{\Omega}$ and it is clear that both Ω and $\bar{\Omega}$ cannot be convex together.

3. PARTICULAR FORMS OF THE CONSTITUTIVE RELATIONSHIP

Specifications of the state variable S_l gives further restriction on the constitutive relationship (11) manifested in the form of the dependence of Ω on the deviatoric stress σ'_{ij} and S_l. We consider three special cases, involving a single state variable, corresponding to isotropic and kinematic strain hardening and work hardening.

(a) *Isotropic strain-hardening*

We define a single state variable as some function of an effective strain $\bar{v}$:

$$S = l(\bar{v}) \tag{21}$$

where

$$\bar{v} = \int_0^t \psi(\dot{v}_{ij})\mathrm{d}t \tag{22}$$

where ψ is a homogeneous scalar function of degree one. Further Ω is assumed to be a function of $\phi(\sigma_{ij})$ where ϕ is a homogeneous of degree one. Substituting these conditions into the constitutive relationships (11) and eliminating v_{ij} yields,

$$A\frac{\partial \Omega}{\partial \phi} = -\frac{1}{l'}\frac{\partial \Omega}{\partial S}, \quad A = \psi\left(\frac{\partial \phi}{\partial \sigma'_{ij}}\right). \tag{23}$$

We term conjugate those ψ and ϕ for which A is a constant, which we may take as unity without loss of generality. For example

$$\phi = \sqrt{\left(\frac{3}{2}\sigma'_{ij}\sigma'_{ij}\right)}, \quad \psi = \sqrt{\left(\frac{2}{3}\dot{v}_{ij}\dot{v}_{ij}\right)} \tag{24}$$

$$\phi = \sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}, \quad \psi = \sqrt{\left(\frac{2}{3}(\dot{v}_{\mathrm{I}}^2 + \dot{v}_{\mathrm{II}}^2 + \dot{v}_{\mathrm{III}}^2)\right)} \tag{25}$$

form conjugate pairs.

The general solution of (23) is given by

$$\Omega = \Omega(\phi - g(\bar{v})), \quad g(\bar{v}) = \int_\phi^{\bar{v}} l'^2\,\mathrm{d}\bar{v}, \quad S = l(\bar{v}) = \int (g')^{1/2}\mathrm{d}\bar{v} \tag{26}$$

and the constitutive relationship (11a) and (11b) reduce to:

$$\dot{v}_{ij} = \Omega'\frac{\partial \phi}{\partial \sigma'_{ij}}, \tag{27}$$

$$\psi(\dot{v}_{ij}) = \Omega'\left(\phi - g\left(\int \psi \mathrm{d}t\right)\right). \tag{28}$$

This general form corresponds to a generalization of a Malvern material [8], and g may be interpreted as a measure of the history dependent "static" flow stress which increases with increasing strain.

The dual potential $\bar{\Omega}$ may be most easily derived by first assuming $\bar{v} = 0$, and hence

$$\Omega(\phi) + \Omega^*(\psi) = \sigma_{ij}\dot{v}_{ij} = \phi\psi. \tag{29}$$

Substituting $(\phi - g)$ for ϕ in (29) and noting (17) we obtain

$$\bar{\Omega}(\dot{v}_{ij}, \bar{v}) = \Omega^*(\psi) + g(\bar{v})\psi. \tag{30}$$

The sufficient conditions for the convexity of Ω, inequalities (12) and (13), may be translated into restrictions on the general function Ω and g:

$$d\sigma'_{ij} d\dot{v}_{ij} = \Omega''(d\phi)^2 \geq o \tag{31}$$

$$dS d\dot{S} = \frac{1}{2} g''\Omega' - g^{-2}\Omega'' \leq 0 \tag{32}$$

where use has been made of eqns (21), (22) and (28). Hence sufficient conditions for the convexity of Ω are given by

$$\Omega' \geq 0, \quad \Omega'' \geq 0, \quad g' \geq 0 \text{ and } g'' \leq 0, \tag{33}$$

and are equivalent to the conditon that $\Omega(\phi)$ shall be convex and $g(\bar{v})$ shall be concave functions of their arguments. In fact, the convexity of Ω may be derived directly from these conditons and a direct and simple proof is given in the appendix.

The meaning of these inequalities in terms of uniaxial behavior of the model may be seen from Figs. 1 and 2. Stress strain curves at constant strain rates must have positive slope ($g' \geq 0$) but must decrease with increasing strain ($g'' \leq 0$). This condition is usually observed in metals and alloys, but mild steel provides a notable exception. On the other hand, stress–strain rate curves at constant strain must have positive slope ($\Omega'' \geq 0$) but the slope may either increase or decrease with increasing strain rate. Hence the inequalities (33) place a more significant restriction on the strain hardening characteristics than on the strain rate sensitivity of the material.

A particularly simple form of Ω which contains a sufficient number of parameters to allow the description of many materials over, at least, some range of stress-strain and strain rate is given by;

$$\begin{aligned} \Omega(\phi - g) &= \frac{\dot{v}_0}{\bar{\sigma}^n(n+1)}\left[\phi - \sigma_0 - \bar{\sigma}\left[\frac{\bar{v}}{v_0}\right]^{1/m}\right]^{n+1}, \quad \phi > \sigma_0 + \bar{\sigma}\left[\frac{\bar{v}}{v_0}\right]^{1/m} \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad , \quad \phi \leq \sigma_0 + \bar{\sigma}\left[\frac{\bar{v}}{v_0}\right]^{1/m} \end{aligned} \tag{34}$$

which gives rise to the uniaxial relationship;

$$\frac{\dot{v}_{11}}{\dot{v}_0} = \left[\frac{\sigma_{11} - \sigma_0}{\bar{\sigma}} - \left[\frac{v_{11}}{v_0}\right]^{1/m}\right]^n, \qquad \sigma_{11} > \sigma_0 + \bar{\sigma}\left[\frac{v_{11}}{v_0}\right]^{1/m}. \tag{35}$$

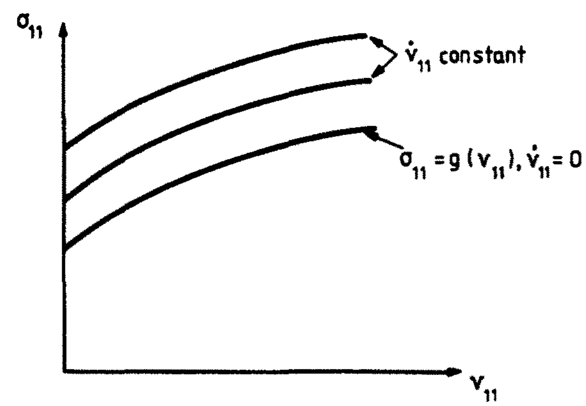

Fig. 1. For convex Ω, stress-strain curves at constant strain rate must have positive slope, decreasing with increasing strain v_{11}.

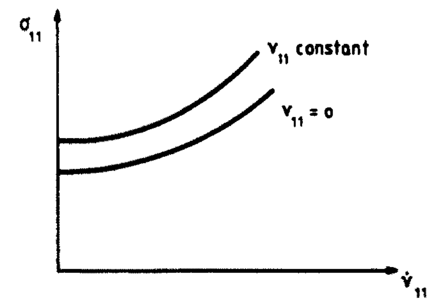

Fig. 2. For convex Ω, stress-strain rate curves at constant strain must have positive slope, but may either increase or decrease with increasing strain rate $\dot{v}_{11}$.

The "static" stress-strain curve ($\dot{v}_{11} = 0$) is given by

$$\frac{v_{11}}{v_0} = \left[\frac{\sigma_{11} - \sigma_0}{\bar{\sigma}}\right]^m, \tag{36}$$

and $n > 0$, $m > 1$ are sufficient condition for the convexity of Ω.

A simple relationship, without an initial yield stress is provided by

$$\begin{aligned} \Omega(\phi - g) &= \frac{\dot{v}_0}{(p+1)\sigma_0{}^p}\left[\phi - \sigma_0\left[\frac{\bar{v}}{v_0}\right]^{1/q}\right]^{p+1}, \qquad \phi > \sigma_0\left[\frac{\bar{v}}{v_0}\right]^{1/q} \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad , \quad \phi \le \sigma_0\left[\frac{\bar{v}}{v_0}\right]^{1/q} \end{aligned}$$

i.e.

$$\frac{\dot{v}_{11}}{\dot{v}_0} = \left[\frac{\sigma_{11}}{\sigma_0} - \left[\frac{v_{11}}{v_0}\right]^{1/q}\right]^p; \quad p > 0, \quad q > 0. \tag{38}$$

Figure 3 shows the correlation of both (35) and (38) with data for 304 stainless steel at 70°F, from the data of Hauser[9]. Clearly eqn (35) provides the better result, but the more restricted form (38) provides a quite satisfactory fit to the data.

For both eqns (34) and (38) the state variable S may be derived from (26):

$$S = \left[\frac{\bar{\sigma}}{m v_0{}^{1/m}}\right]^{1/2}\left(\frac{2m}{m+1}\right)\bar{v}^{(1-m)/2m},$$

where $m = q$ and $\bar{\sigma} = \sigma_0$ for eqn (38).

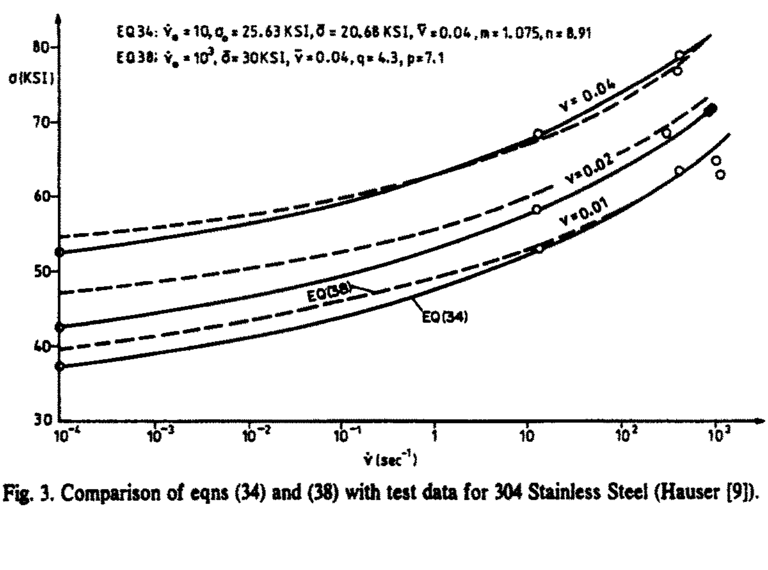

Fig. 3. Comparison of eqns (34) and (38) with test data for 304 Stainless Steel (Hauser [9]).

(b) *Kinematic—strain hardening*

Kinematic behavior may be generated by assuming S_l are the components of a second order tensor quantity S_{ij}. The simplest form occurs when

$$S_{ij} = C\, v_{ij}$$

where C denotes a constant. The state equations then yield that

$$\Omega = \Omega(\sigma'_{ij} - C\, v_{ij}) \tag{39}$$

and Ω is convex, provided $\Omega'' \geq 0$ and $C > 0$.

(c) *Work hardening*

Constitutive equations in which the state is given by the plastic work have been discussed by Perzena[10] and Bodner and Partom[11]. If in eqn (11) we choose

$$S = l(W), \quad W = \int_0^t \sigma_{ij}\dot{v}_{ij}\mathrm{d}t, \tag{40}$$

and assume that $\Omega = \Omega(\phi(\sigma_{ij}), S)$, then eqns (11) yield, on eliminating $\dot{v}$,

$$\phi\frac{\partial \Omega}{\partial \phi} + \frac{1}{l'}\frac{\partial \Omega}{\partial l} = 0 \tag{41}$$

which possess the general solution;

$$\Omega = \Omega(\phi \mathrm{e}^{-g(W)}) \text{ where } g(W) = \int (l')^2 \mathrm{d}W. \tag{42}$$

The strain rate equation is now given by

$$\dot{v}_{ij} = \Omega'(\phi \mathrm{e}^{-g(W)})\frac{\partial \phi}{\partial \sigma'_{ij}} \cdot \mathrm{e}^{-g(W)}. \tag{43}$$

This equation is of a similar form to that suggested by Bodner and Partom [11], but their equation cannot be expressed in terms of a potential of the form (42). As (42) and (43) appear to be essentially new, their properties will not be pursued further in any detail. It is worth noting, however, that sufficient conditions for convexity of Ω are given by the restriction that Ω shall be a convex function of ϕ and a concave function of S. The resulting restriction becomes a combined restriction on the form of Ω and $g(w)$ and will not be pursued here.

4. THE CONVERGENCE THEOREM

Consider a body of volume V surface $\mathscr{S}$ subjected to an impulse at time $t = 0$ which gives rise to an initial velocity distribution $\dot{u}_i(0)$. Surface traction T_i act over part of $\mathscr{S}$, $\mathscr{S}_T$ and body force $F_i(t)$ act within V. Over the remainder of $\mathscr{S}$, which we denote by $\mathscr{S}_u$, displacements u_i remain zero.

The theorem concerns two such bodies subjected to identical conditions in all respects except the conditions at time $t = 0$. One body has initial stress σ^1_{ij}, velocity $\dot{u}^1$ and state S^1_l, whereas the second body has differing initial value σ^2_{ij}, $\dot{u}^2_{ij}$, S^2_l. We are concerned with the relationship between the subsequent histories in the two bodies.

We first note the following relationship, which follow from the constitutive relationship (9) and (10) and the principle of virtual velocities:

$$\int_V (\sigma_0{}^1 - \sigma^2_{ij})(\dot{\epsilon}^1_{ij} - \dot{\epsilon}^2_{ij})\mathrm{d}\bar{v} = -\frac{\mathrm{d}}{\mathrm{d}t}\Delta_v[u_i^1 - u_i^2] = \int_V (\sigma^1_{ij} - \sigma^2_{ij})(\dot{v}^1_{ij} - \dot{v}^2_{ij})\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\Delta_e[\sigma^1_{ij} - \sigma^2_{ij}] \tag{44}$$

where

$$\Delta_v[\dot{u}_i^1 - \dot{u}_i^2] = \int_V \frac{1}{2}\rho(\dot{u}_i^1 - \dot{u}_i^2)(\dot{u}_i^1 - \dot{u}_i^2)\mathrm{d}V \tag{45}$$

and

$$\Delta_e[\sigma_{ij}^1 - \sigma_{ij}^2] = \int_V \frac{1}{2} C_{ijkl}(\sigma_{ij}^1 - \sigma_{ij}^2)(\sigma_{kl}^1 - \sigma_{kl}^2)\mathrm{d}V. \tag{46}$$

Hence $\Delta_v(\dot{u}_i)$ denotes the total kinetic energy associated with velocity distribution $\dot{u}_i$ and $\Delta_e(\sigma_{ij})$ denotes the total strain energy associated with stress distribution σ_{ij}.

Combining (44) with the convexity condition (15) yields

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} \leq 0 \tag{47}$$

where

$$\Delta = \Delta_v(\dot{u}_i^1 - \dot{u}_i^2) + \Delta_e(\sigma_{ij}^1 - \sigma_{ij}^2) + \Delta_s(S_l^1 - S_l^2), \tag{48}$$

and

$$\Delta_s = \int_V \frac{1}{2}(S_k^1 - S_k^2)(S_k^1 - S_k^2)\mathrm{d}V.$$

Further

$$\Delta \geq 0. \tag{49}$$

Equality in (47) occurs only when $\sigma_{ij}^1 = \sigma_{ij}^2$ and $S_l^1 = S_l^2$ for strictly convex Ω and when $\dot{u}_1 = \dot{u}_2$. Equality occurs in (49) under the same conditions. Hence σ_{ij}^1, u_{ij}^1 and S_l^1 approach σ_{ij}^2, u_{ij}^2 and S_l^2 in the sense that Δ must reduce in magnitude. It is clear that either Δ_e or Δ_v or Δ_s may increase in value during the deformation process. Indeed, if $S_l^1 = S_l^2$ at $t = 0$, initially Δ_s will certainly increase but less rapidly than the combined decrease of Δ_v and Δ_e.

Our principal interest will be in the case where the electric term Δ_e is negligible and initially $S_l^1 = S_l^2$. At $t = 0$ Δ_v must desrease in time and will continue to do so until Δ_s has become of a similar magnitude to Δ_v. The convergence may be expected to be initially dominated by u_i^1 approaching u_i^2 and eventually S_l^1 approaching S_l^2 when $\dot{\Delta}_v$ is sufficiently small. How this behavior manifests itself in the behavior of a simple structure is demostrated in the section.

5. AN EXAMPLE

Consider the structural model shown in Fig. 4. Two masses are attached to weightless beam of length l which is simply supported at each end.

The kinematics of the structure may be expressed in terms of the vertical displacements

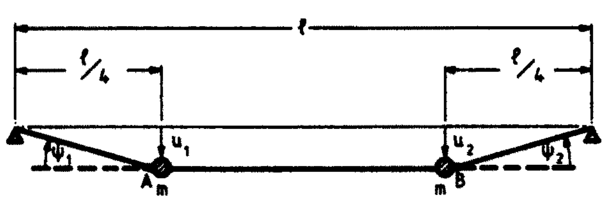

Fig. 4. Simple structural model considered as an example.

$U_1(t)$ and $U_2(t)$ of the two masses. Deformation is assumed to be confined to points A and B where hinges form which rotate with angles ψ_1 and ψ_2 given by

$$\psi_1 = \frac{2}{l}(3U_2 - U_2), \quad \psi_2 = \frac{2}{l}(3U_2 - U_1). \tag{50}$$

The moments applied to these hinges are given by

$$M_1 = \frac{l}{16}(3P_1 + P_2), \quad M_2 = \frac{l}{16}(3P_2 + P_1) \tag{51}$$

where P_1 and P_2 are the D'Alembert forces applied by the weights to the beam and hence

$$m\ddot{U}_1 = -P_1, \quad m\ddot{U}_2 = -P_2. \tag{52}$$

For the relationship between ψ_1 and M_1 we adopt a simple relationship of the form of eqn (35):

$$\dot{\psi}_i = \dot{\psi}_0 \left[\frac{|M_i|}{M_o} - 1 - \left(\frac{\bar{\psi}_i}{\psi_o}\right)^{1/m}\right]^n \langle M_i \rangle \tag{53}$$

where $\bar{\psi}_i = \int_0^t |\dot{\psi}_i| \, \mathrm{d}t$.

Equations (50)–(53) may be combined into a pair of second order differential equations for U_i. For $\dot{\psi} > 0$ and $\dot{\psi} > 0$, the equation takes the form

$$\ddot{u}_i = \frac{1}{8^{[(n+1)/n]}}[3(3\dot{u}_1 - \dot{u}_2)^{1/n} - (3\dot{u}_2 - \dot{u}_1)^{1/n} + 8^{1/n}(2 + 3\beta_1 - \beta_2)] \tag{54}$$

$$\ddot{u}_2 = \frac{1}{8^{[(n+1)/n]}}[(3\dot{u}_1 - \dot{u}_2)^{1/n} - 3(3\dot{u}_2 - \dot{u}_1)^{1/n} + 8^{1/n}(-2 + \beta_1 - 3\beta_2)] \tag{55}$$

and similar equations hold for differing sign combinations of ψ_i. The equations are expressed in terms of non-dimensional variables:

$$\dot{u}_i = 16\frac{\dot{U}_i}{\dot{\psi}_0 l}, \quad p_i = \frac{P_i l}{16\, M_o},$$

and

$$\beta_1 = A^{1/m}\left[\int_0^\tau |3\dot{u}_1 - \dot{u}_2| \, \mathrm{d}\tau'\right]^{1/m},$$

$$\beta_2 = A^{1/m}\left[\int_o^\tau |3\dot{u}_2 - \dot{u}_1| \, \mathrm{d}\tau'\right]^{1/m},$$

$$\tau = \frac{256\, M_o t}{m\dot{\psi}_o l^2}, \quad \text{and} \quad A = \frac{m\dot{\psi}_o^2 l^2}{256.8\, M_o \psi_o}.$$

In terms of these non-dimensional quantities the solution is governed by $\dot{u}_i$ at $t = 0$, and the material parameter m, n and A. The quantity A governs the degree of strain hardening in the model as $A = 0$ corresponds to purely viscous behavior.

We may compute values of A which have some resemblences to reality by the following approximate argument. We may identify rotation rate for the special case $u_1 = \dot{u}_2$. The initial kinetic energy of the structure is then given by

$$K_o = \frac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2) = \frac{ml}{16}\dot{\psi}_0^2.$$

With $\bar{\psi}_i = 0$ and $\dot{\psi}_i = \psi_0$ then $M_i = 2\,M_o$ where M_o is the static initial yield moment. Hence this choice of initial velocities corresponds to a doubling of the initial yield stress. We may now write;

$$A = \frac{K_o}{64\,W} \quad \text{where} \quad W = 2\,M_O\psi_o.$$

If $\psi_i = 0$ and $\dot{\psi}_i = \psi_0$ then again $M = 2\,M_o$, a degree of strain hardening which is similar in magnitude to that exhibited by the data of Fig. 2. Hence, for this hypothetical case, assuming ψ_0 is the final rotation of the hinges the moments M_i begin and end at the same value, $2\,M_0$. If we assume that on average M_i remain close to this value, the total energy dissipated in the hinges is given by $4\,M_0\psi_0$ and equals the initial kinetic energy K. Hence, $A = 1/32$ and this value may be expected to yield an upper limit on a reasonable range of A.

With $\dot{\psi}_1 = \dot{\psi}_2 = \dot{\psi}_0$, then the non-dimensional velocities at $\tau = 0$ are $\dot{u}_1 = \dot{u}_2 = 4$. In the following numerical examples, we choose initial velocities which yield the same values of the non-dimensional kinetic energy as $\dot{u}_1 = \dot{u}_2 = 4$.

The eqns (54) and (55) were solved numerically by treating them as first order equations in $\dot{u}_i$ and integrating to compute β_i. The time interval was reduced until the solution showed no appreciable change in the solution. Cases were run for values of m for which the theorem indicated convergence occured ($m \geq 1$) and also for cases where the theorem was not valid ($m < 1$) to see if non-convergence could be observed.

Two combinations of the exponents n and m were studied. In all cases the value $n = 5$ was chosen and solutions were computed for a range of values of A for $m = 3$ and $m = 1/3$. In the case $m = 3$, the conditions of the theorem were satisfied and convergence was expected, whereas for $m = 1/3$ the conditions of the theorem were not satiafied and convergence may or may not occur. In fact, we discovered that in this later case, convergence occurred for sufficiently small values of A but divergence occured for larger values of A.

The theorem was tested by evaluating the two solutions corresponding to $\dot{u}_2 = 0$ and $\dot{u}_1 = \dot{u}_2/3$, such that their initial kinetic energy were the same. For $m = 3$ the solutions for increasing A are very similar in form and closely follow the $A = 0$ solutions. In Fig. 5 the displacements (u_1, u_2) are shown and we see that all histories follow similar paths, except that the final displacements are markedly affected by the value of A. For $A = 0.3$ the final deflections are approximately one-half the values for $A = 0$. The velocity trajectories are shown in Fig. 6. Strain hardening appears to have little effect on either the velocity ratio $\dot{u}_1/\dot{u}_2$ or the displacement shape, but causes the structure to come to rest more quickly and with a smaller deflection.

The values of the convergence quantity Δ/K_0 with τ is shown in Fig. 7 for the extreme cases $A = 0$ and $A = 0.5$. For $A = 0$, $\Delta = \Delta_v$ decreases rapidly until $\tau \simeq 1.5$ when it continues to decline very slowly. For $A = 0.5$, Δ_v decreases even more quickly and, in fact, decreases throughout

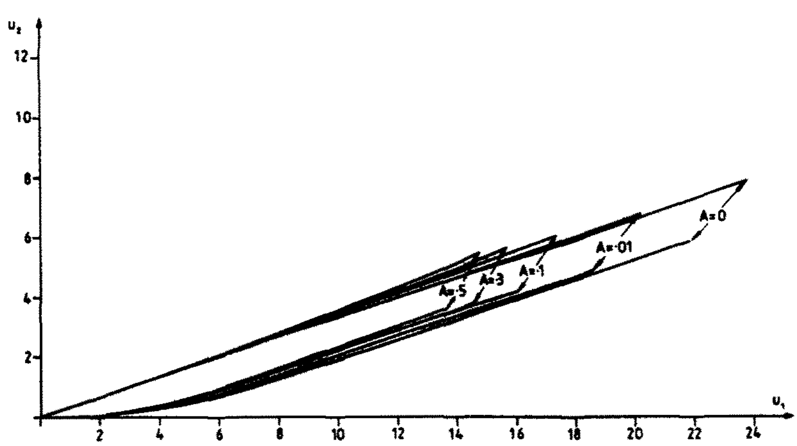

Fig. 5. Non-dimensional displacements (u_1, u_2) for $n = 5$ and $m = 3$ for a range of values of the non-dimensional material strain-hardening parameter A.

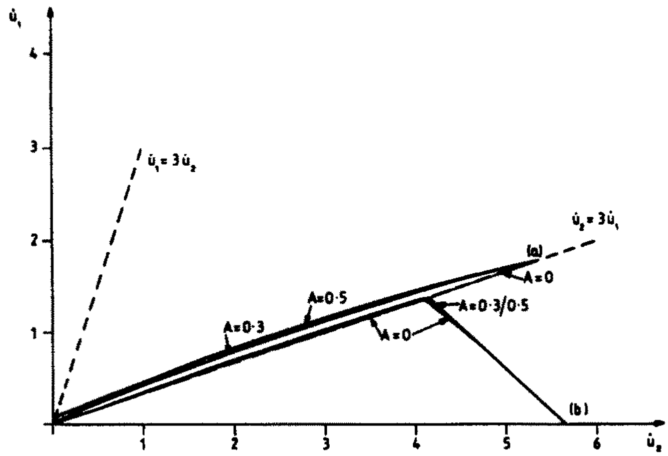

Fig. 6. Non-dimensional velocity paths for $n = 5$, $m = 3$ and a range of values of A.

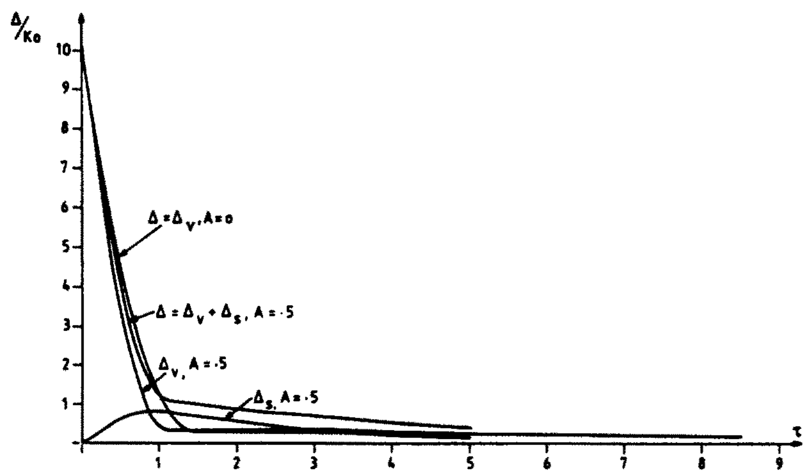

Fig. 7. Variation of convergance quantity $\Delta = \Delta_v + \Delta_s$ for $n = 5$ and $m = 3$.

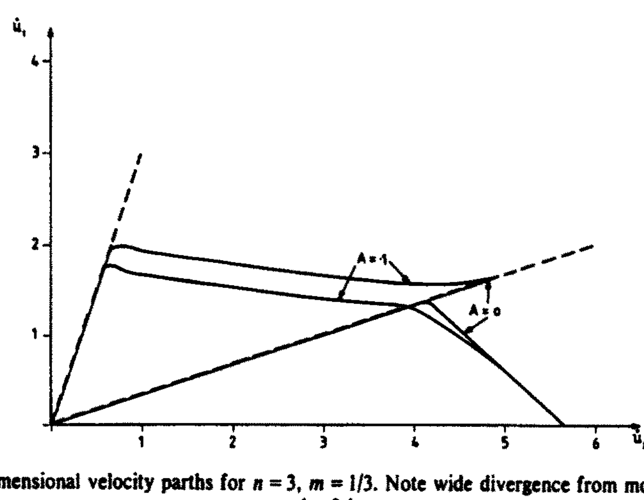

Fig. 8. Non-dimensional velocity parths for $n = 3$, $m = 1/3$. Note wide divergence from mode solution for $A = 0.1$.

the history of the structure. Δ_s increases to a maximum value of less than 10% of $\Delta(0)$ before decreasing in value. Hence Δ_s remains very small and Δ is dominated by Δ_v. It should be emphasized that in this example, the effect of strain hardening is sufficiently great to reduce the final displacement by a factor of two and to reduce the deformation time by a similar order of magnitude.

These results imply that for moderate amounts of strain hardening the displacement shape and velocity distribution are relatively insensitive to the amount of strain hardening, but the total deflection and response time are markedly affected. This phenomenon may be explained by the notion of mode solutions which will be discussed in a forthcoming paper.

For the case of $m = 1/3$, we find that convergence occurs for sufficiently small values of A as the structure is brought to rest before the strain hardening terms are of significant magnitude. The velocity histories for $A = 0.01$ and 0.1 are shown in Fig. 8 and the displacements and Δ's are shown in Figs. 9 and 10. For $A = 0.01$ the solutions are very close to $A = 0$ solutions.

For $A = 0.1$, however, the solutions show non-convergent properties and have distinctly differing properties from those for $m = 3$. In Fig. 8 it can be seen that the velocity paths of both solutions initially follow the parts corresponding to $A = 0$ and subsesuently move over to and remain close to the line $\dot{u}_2 = 3\dot{u}_1$ (i.e. $\dot{\psi}_1 = 0$). This behaviour is reflected in the displacement paths shown in Fig. 9 and imply that the total history consists of two parts. During the initial part $\dot{\psi}_2$ remains small and during the final part $\dot{\psi}_1$ remains small. This type of behaviour contrasts strongly with mode solutions where the displacement form remains constant in time. The fact that convergence has not occurred can be clearly seen in Fig. 10 where Δ_v, Δ_s and Δ both increase and decrease during the course of the deformation history.

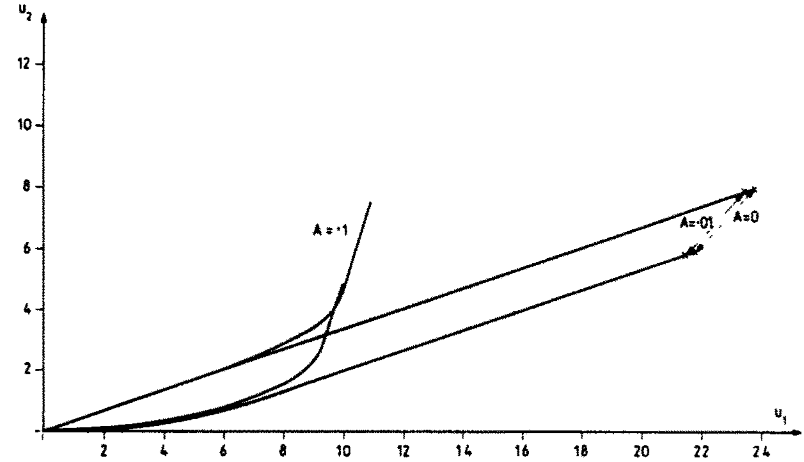

Fig. 9. Non-dimensional displacements (u_1, u_2) for $n = 3$ and $m = 1/3$ for a range of values of the non-dimensional material strain hardening parameter A.

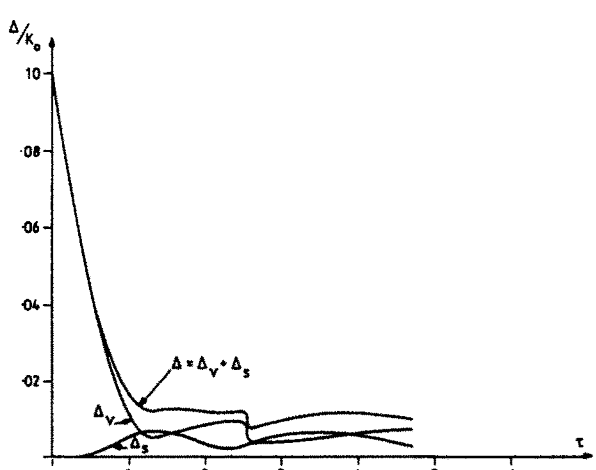

Fig. 10. Variation of convergence quantity $\Delta = \Delta_v + \Delta_s$ for $n = 5$, $m = 1/3$. Note non-convergence for $\tau > 1$.

6. CONCLUSION

A dynamic convergence theorem has been proven for a class of viscoplastic constitutive equations involving internal state variables. The constitutive equations are expressed in terms of a scalar potential function and particular forms are derived by assuming that the state may be calculated from either the plastic strain, the effective plastic strain or the plastic work. For the case of effective plastic strain the form becomes the familiar Malvern material.

For a particular simple model, structure dynamic solutions are generated for cases when the convergence proof holds, and the resulting solutions follow closely the mode-type of solution of a visco-plastic solid. For circumstances when the convergence proof does not hold, non-convergence can be demonstrated, although it is clear that for weak strain hardening (corresponding to small values of the non-dimensional parameter A) convergence can occur as the theorem is not a necessary, but only a sufficient condition for convergence.

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APPENDIX

The inequality (14) may be derived in the case

$$\Omega(\sigma_{ij}, s) = \Omega(\phi(\sigma_{ij}) - g(\bar{v})), \tag{A1}$$

and

$$g(\bar{v}) = \int_0^{\bar{v}} l'^2 \, d\bar{v}, \quad l(\bar{v}) = s, \tag{A2}$$

directly from the conditions (33)

$$\Omega' \geq 0 \tag{A3}$$

$$\Omega'' \geq 0 \tag{A4}$$

$$g' \geq 0 \tag{A5}$$

$$g'' \leq 0 \tag{A6}$$

and the further condition that $\phi(\sigma_{ij})$ shall be convex.

Transforming $g(\bar{v}) = \theta(s)$ where for A2

$$\theta'(s) = g'^{1/2} \geq 0 \tag{A7}$$

and

$$\theta''(s) = \frac{1}{2}\frac{g''}{g'} \leq 0. \tag{A8}$$

Inequalities (A3) and (A4) imply the convexity of Ω

$$\Omega(\phi^1) - \Omega(\phi^2) - \Omega'(\phi^2)(\phi^1 - \phi^2) \geq 0. \tag{A9}$$

Similarity (A7) and (A8) imply the concavity of $\theta(s)$

$$-(\theta(s^1) - \theta(s^2)) + \theta'(s^2)(s^1 - s^2) \geq 0. \tag{A10}$$

If ϕ^1 and ϕ^2 are replaced by $\phi^1 - \theta(s^1)$ and $\phi^2 - \theta(s^2)$ in (A9) and inequality (A10) multiplied by $\Omega'(\phi^2) \geq 0$ is added to the resulting inequality, we obtain

$$\Omega(\phi^1 - \theta^1) - \Omega(\phi^2 - \theta^2) - \Omega'(\phi^2 - \theta^2).[(\phi^1 - \phi^2) - \theta'(s^2)(s^1 - s_2) \geq 0. \tag{A11}$$

On further noting that

$$\phi^1 - \phi^2 \geq \left(\frac{d\phi}{d\sigma_{ij}}\right)_2 (\sigma_{ij}^1 - \sigma_{ij}^2)$$

and that

$$\frac{d\Omega}{d\sigma_{ij}} = \Omega' \frac{d\phi}{d\sigma_{ij}}$$

and

$$\frac{d\Omega}{ds} = -\Omega'\theta'.$$

then (A11) becomes

$$\Omega(\phi^1 - \theta^1) - \Omega(\phi^2 - \theta^2) - \left(\frac{d\Omega}{d\sigma_{ij}}\right)_2 (\sigma_{ij}^1 - \sigma_{ij}^2) - \left(\frac{d\Omega}{ds}\right)_2 (s_1 - s_2) \geq 0$$

as required.